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Interacting Electromagnetic Waves in General Relativity

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Abstract

The problem is considered of finding exact solutions of the Einstein–Maxwell equations which describe the physical situation of two colliding and subsequently interacting electromagnetic waves. The general theory of relativity predicts a nonlinear interaction between electromagnetic waves. The situation is described using an approximate geometrical method, and a new exact solution describing two interacting electromagnetic waves is given. This describes waves emitted from two sources mutually focusing each other on the opposite source.

1. *Introduction*

Electromagnetic waves are described mathematically as solutions of Maxwell's equations. These equations are linear, and solutions can be simply superposed. This means that in Maxwell's model, electromagnetic waves do not interact. However, it is unlikely that any physical theory is exactly linear, and therefore a (usually negligible) interaction may be expected between electromagnetic waves. Such an interaction in fact appears when electromagnetic fields are described in the general theory of relativity. Maxwell's equations remain linear in form, but the gravitational field equations are highly nonlinear. The physical situation may therefore be considered in the following way. Any electromagnetic wave or field is associated with a gravitational wave or field which is coupled to it through the Einstein–Maxwell equations. When any two electromagnetic waves or fields meet, their associated gravitational fields interact nonlinearly causing an interaction between the two fields.

The effects of this mutual interaction of electromagnetic waves will be minute, and it is unlikely that it will ever be directly detected experimentally. However, it may have significance cosmologically. It is possible that photons from very distant galaxies will have suffered sufficient interactions to distort their observation. It is also possible that this effect will have significance on the subatomic scale.

In this work I attempt to consider the consequences of the nonlinearity of Einstein's equations as far as electromagnetic waves are concerned. I attempt to describe the nonlinear interaction between electromagnetic waves that is predicted by the general theory of relativity. The interaction between electromagnetic waves is caused basically by the interaction between their associated gravitational waves, and therefore it is necessary also to consider the related problem of the collision and interaction of gravitational waves.

Obviously both exact and approximate methods need to be considered. The approximate geometrical method described by Penrose (1966) is relevant here. In the following work it is assumed that no fields are present other than those of the electromagnetic or gravitational waves considered.

2. *The Spin Coefficient Formalism*

It is particularly appropriate to develop the techniques of geometrical optics to describe a congruence of null rays in curved space-time. For details of this approach see Jordan *et al.* (1961). However, the notation and methods of Newman and Penrose (1962, 1963) are used here. Into the space-time manifold a tetrad of null vectors $l_\mu, n_\mu, m_\mu, \bar{m}_\mu$ is introduced, l_μ and n_μ being real and future pointing, and m_μ and its complex conjugate $\bar{m}_\mu$ being formed from a pair of unit spacelike vectors a_μ and b_μ orthogonal to l_μ and n_μ by $m_\mu = 2^{-1/2}(a_\mu - ib_\mu)$. Then with a suitable scaling $l_\mu n^\mu = 1$, $m_\mu \bar{m}^\mu = -1$ and the other products are all zero. The spin coefficients are the complex tetrad components of the covariant derivatives of the tetrad vectors. They may be defined by

$$\kappa = l_{\mu;\nu} m^\mu l^\nu, \qquad \nu = -n_{\mu;\nu} \bar{m}^\mu n^\nu$$

$$\rho = l_{\mu;\nu} m^\mu \bar{m}^\nu, \qquad \mu = -n_{\mu;\nu} \bar{m}^\mu m^\nu$$

$$\sigma = l_{\mu;\nu} m^\mu m^\nu, \qquad \lambda = -n_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu$$

$$\tau = l_{\mu;\nu} m^\mu n^\nu, \qquad \pi = -n_{\mu;\nu} \bar{m}^\mu l^\nu$$

$$\epsilon = \tfrac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \bar{m}^\mu l^\nu)$$

$$\alpha = \tfrac{1}{2}(l_{\mu;\nu} n^\mu \bar{m}^\nu - m_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu)$$

$$\beta = \tfrac{1}{2}(l_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu} \bar{m}^\mu m^\nu)$$

$$\gamma = \tfrac{1}{2}(l_{\mu;\nu} n^\mu n^\nu - m_{\mu;\nu} \bar{m}^\mu n^\nu)$$

The spin coefficients are related through the tetrad to the geometrical properties of the manifold. For example, if $\kappa = 0$ then l_μ is tangent to a geodesic null congruence. If in addition the parameter along the congruence is affine then $\mathrm{Re}\,\epsilon = 0$ and $\mathrm{Re}\,\rho$, $\mathrm{Im}\,\rho$, and σ define the contraction, twist, and shear of the congruence, respectively. In the congruence defined by n_μ the coefficients $-\nu$, $-\mu$, $-\lambda$ correspond to κ, ρ, σ, respectively.

The ten independent components of the Weyl tensor $C_{\kappa\lambda\mu\nu}$ may be defined conveniently in terms of five independent complex tetrad components by

$$\Psi_0 = -C_{\kappa\lambda\mu\nu} l^\kappa m^\lambda l^\mu m^\nu$$
$$\Psi_1 = -C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda l^\mu m^\nu$$
$$\Psi_2 = -\tfrac{1}{2} C_{\kappa\lambda\mu\nu} l^\kappa n^\lambda (l^\mu n^\mu - m^\mu \bar{m}^\mu)$$
$$\Psi_3 = -C_{\kappa\lambda\mu\nu} n^\kappa l^\lambda n^\mu \bar{m}^\nu$$
$$\Psi_4 = -C_{\kappa\lambda\mu\nu} n^\kappa \bar{m}^\lambda n^\mu \bar{m}^\nu$$

These components have the following physical meaning (Szekeres, 1965): Ψ_0 and Ψ_1 (or Ψ_4 and Ψ_3) describes transverse and longitudinal gravitational wave components in the n_μ (or l_μ) direction, respectively. Ψ_2 denotes a coulomb component.

The tetrad components of the electromagnetic field tensor $F_{\mu\nu}$ are denoted by

$$\Phi_0 = F_{\mu\nu} l^\mu m^\nu, \qquad \Phi_1 = \tfrac{1}{2} F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu), \qquad \Phi_2 = F_{\mu\nu} \bar{m}^\mu n^\nu$$

where Φ_0 and Φ_2 describe null wave components in the n_μ and l_μ directions, respectively. The components of the Ricci tensor are now given in terms of these components, and the gravitational field equations become a set of equations involving the first derivatives of the spin coefficients (Newman and Penrose, 1962; Trim, 1972). The first two equations upon which attention is concentrated in following sections are

$$D\rho - \bar{\delta}\kappa = \rho^2 + \sigma\bar{\sigma} + \rho(\epsilon + \bar{\epsilon}) - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_0\bar{\Phi}_0$$
$$D\sigma - \delta\kappa = \sigma(\rho + \bar{\rho}) + \sigma(3\epsilon - \bar{\epsilon}) - \kappa(\tau - \bar{\pi} + \bar{\alpha} + 3\beta) + \Psi_0$$

where $D\phi = \phi_{;\mu} l^\mu$, $\delta\phi = \phi_{;\mu} m^\mu$, etc.

3. *Approximate Geometrical Method*

Consider a geodesic congruence of null rays and align the vector l_μ with it. Then $\kappa = 0$ and it is possible to scale the tetrad so that in addition $\epsilon = 0$. The first two equations then read

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_0\bar{\Phi}_0 \tag{3.1}$$

$$D\sigma = \sigma(\rho + \bar{\rho}) + \Psi_0 \tag{3.2}$$

If there is no gravitational or electromagnetic radiation with a component in direction different from $l_\mu(\Phi_0 = 0, \Psi_0 = 0)$, then it is possible for the rays to be parallel (zero contraction, twist, and shear, $\rho = 0$, $\sigma = 0$). If such rays then meet an electromagnetic wave with $\Phi_0 \neq 0$ and $\Psi_0 = 0$, then the rays will start to contract and a singularity like a focus will occur a finite distance along the rays. Alternatively, if the rays meet a transverse gravitational wave ($\Psi_0 \neq 0$, $\Phi_0 = 0$), then the rays will start to shear and the term $\sigma\bar{\sigma}$ in (1) will also cause them to contract. Thus the gravitational wave causes the rays to focus astigmatically. These properties have been considered in detail by Penrose (1966).

It has been suggested by Szekeres (1972) that such mutual focusing of gravitational and electromagnetic waves could produce the unexpectedly high energy of the gravitational waves observed by Weber (1969). However, these observations have still not been confirmed (Papini, 1974).

4. *Colliding Gravitational and Electromagnetic Waves*

The effects of the collision and interaction of two gravitational or electromagnetic waves are now considered. The null vector l_μ can be aligned with one wave and n_μ with the other. This is equivalent to making a Lorentz transformation to a frame of reference in which the two waves approach each other from exactly opposite spatial directions. It is therefore necessary only to consider "head-on" collisions. Now consider two colliding plane electromagnetic waves. A collision of two such sandwich waves is pictured in Figure 1.

The situation described can be seen to be essentially a boundary value problem with initial data given along the two null hypersurfaces indicated by thick lines in Figure 1. Such a situation in the vacuum case has been considered by Sachs (1962). Robson (1973) has shown that the appropriate junction conditions are those proposed by O'Brien and Synge (1952). Bell and Szekeres (1974) have recently confirmed this by showing that a discontinuity in the normal derivative must occur in colliding electromagnetic waves, so that the familiar Lichnerowicz conditions must necessarily be relaxed.

Exact solutions describing colliding gravitational waves have been obtained

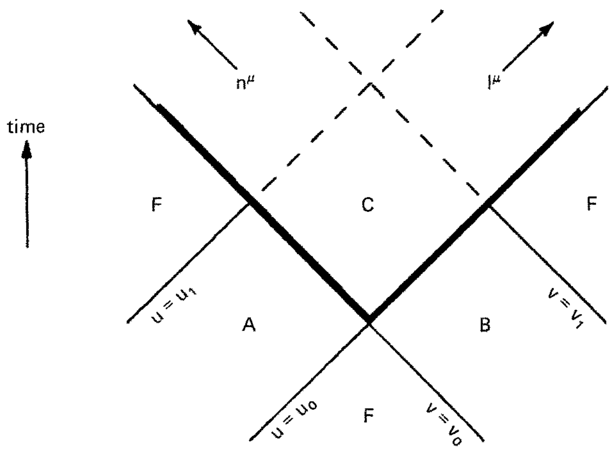

Figure 1—The regions F are flat, and A and B represent two sandwich waves propagating in opposite directions. C is the interaction region. Null coordinates u and v along the waves are chosen for convenience.

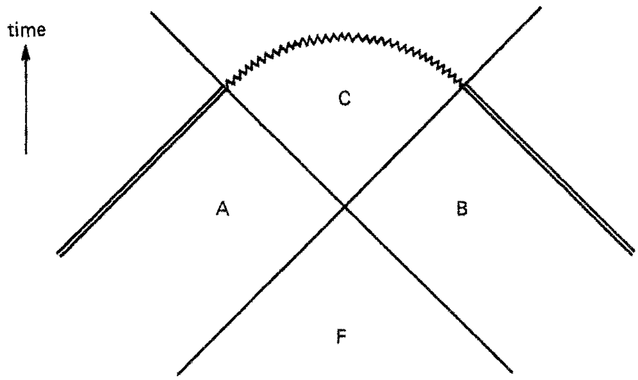

Figure 2—Colliding plane gravitational waves. The double lines represent the coordinate singularities in regions A and B which are contained in the Rosen metric. The jagged line represents the singularity which occurs in the interaction region.

by Khan and Penrose (1971) and Szekeres (1970, 1972). The structure of these solutions is shown in Figure 2. These solutions are given in terms of the Rosen (1937) metric

$$ds^2 = 2e^{-M}\, du\, dv - e^{-U}(e^V\, dx^2 + e^{-V} dy^2)$$

where M, U, and V are functions of both u and v in the interaction region. The singularities in the metric may be interpreted as being caused by the focusing of geodesic congruences across the waves. The singularity found in the interaction region is an essential singularity and along it Ψ_0, Ψ_2, and Ψ_4 are all infinite. The remaining are removable coordinate singularities.

An exact solution describing colliding electromagnetic waves has been obtained by Bell and Szekeres (1974), again in terms of the Rosen metric. However, in this solution the fields inside all regions are conformally flat. The singularity in the interaction region is a nonessential coordinate singularity and it can be seen that in the entire region Ψ_A are zero and Φ_0 and Φ_2 constants. The singularities in this solution may be regarded as being caused by the focusing of geodesics across the waves, but the field at the focal points remains bounded. This is therefore a very restricted type of solution and it may be expected that a collision of more general (i.e., not conformally flat) electromagnetic waves will result in essential singularities.

5. *The Interaction Region*

Although the physical situation being considered corresponds essentially to a boundary value problem with initial data given on two null hypersurfaces, it is very difficult to obtain exact solutions and the following alternative approach is instructive. This concentrates attention on the interaction region. The approach basically is to guess the nature of the field in the interaction region and to look for an exact solution. Only when an exact solution is obtained is it necessary to look for suitable boundary conditions giving rise to it. Thus asymptotic properties need not be considered when looking for the exact solution. This approach has already been applied successfully (Griffiths, 1975a,) to a study of interacting gravitational waves, although suitable boundary conditions for the exact solution obtained have not yet been found.

Now consider a region containing two null electromagnetic waves propagating in different directions. If the tetrad is aligned with the two waves then $\Phi_1 = 0$. The source-free Maxwell equations (Newman and Penrose, 1962) then become

$$D\Phi_2 = (\rho - 2\epsilon)\Phi_2 - \lambda\Phi_0$$

$$\delta\Phi_2 = (\tau - 2\beta)\Phi_2 - \nu\Phi_0$$

$$\Delta\Phi_0 = -(\mu - 2\gamma)\Phi_0 + \sigma\Phi_2$$

$$\bar{\delta}\Phi_0 = -(\pi - 2\alpha)\Phi_0 + \kappa\Phi_2$$

It can be seen from these equations that when two electromagnetic waves interact, they no longer necessarily follow shear-free null geodesic congruences as they do in the noninteraction regions (Mariot, 1954; Robinson, 1961). If the rays are shearing, and particularly if they are nongeodesic, it is very difficult to solve the field equations, since the spin coefficient method of Newman and Penrose provides no simplifications. However, it is reasonable to assume that there is no electromagnetic interaction between the two waves. Thus it would be expected that both waves would independently be solutions of Maxwell's equations. If this were the case, both congruences would continue to be geodesic and shear-free in the interaction region. There is as yet only one known exact solution corresponding to this case (Griffiths, 1976), but suitable boundary conditions for it have not yet been obtained.

Szekeres has pointed out to me that the above criterion for the independence of the two waves may in fact be a trifle strong. In his exact solution (Bell and Szekeres, 1974), the interacting rays are geodesic but have nonzero shear. Since this is an exact global solution it must be concluded that there may be an electromagnetic interaction between two waves given through Maxwell's equations. In the new exact solution given in Section 6, the interacting rays are similarly geodesic and shearing.

Even in the case where the two waves follow shear-free geodesic null congruences, it is still difficult to solve the field equations on account of the term $\Phi_0\bar{\Phi}_0$ that remains in equation (3.1). This in effect prevents the integration to obtain ρ. However $\Phi_0\Phi_2 - \Phi_1{}^2$ is an invariant in Einstein-Maxwell theory (Debney and Zund, 1971), and it is possible to make a transformation to a new tetrad for which Φ_0 and Φ_2 are simultaneously zero. Such a transformation is

$$l_\mu = \frac{1}{2}\left(\frac{\Phi_{00}}{\Phi_{22}}\right)^{1/4} (l'_\mu - im'_\mu + i\bar{m}'_\mu + n'_\mu)$$

$$m_\mu = \frac{1}{2}\left(\frac{\Phi_{02}}{\Phi_{20}}\right)^{1/4} (-il'_\mu + m'_\mu + \bar{m}'_\mu + in'_\mu)$$

$$n_\mu = \frac{1}{2}\left(\frac{\Phi_{22}}{\Phi_{00}}\right)^{1/4} (l'_\mu + im'_\mu - i\bar{m}'_\mu + n'_\mu)$$

Then $\Phi'_1 = -i(\Phi_0\Phi_2)^{1/2}$, and the condition for two independent electromagnetic waves

$$\kappa = \sigma = \nu = \lambda = 0 \tag{5.1}$$

becomes

$$\kappa' = \pi' + 2\alpha', \qquad \sigma' = \mu' + 2\gamma', \qquad \nu' = \tau' + 2\beta', \qquad \lambda' = \rho' + 2\epsilon'$$

It will, however, still be difficult to find a solution of the field equations unless one of the principal null vectors is geodesic and shear-free. But, if this extra condition is added, it is possible to show after a tedious calculation that no exact solutions exist in this case. The above transformation will prove useful, however, in the case in which the conditions (5.1) are relaxed. In fact all exact solutions of the Einstein-Maxwell equations for which $\Phi_0 = 0$, $\Phi_1 \neq 0$, $\Phi_2 = 0$ can be transformed back into solutions of the type required. An example of such a solution is that given by Tariq and Tupper (1974). Other examples may be obtained from the results of Trim (1972).

If the assumption (5.1) is made, both congruences are geodesic and shear-free. It is now necessary to consider the contraction (or expansion) and twist of the congruences. In this approximate approach Penrose (1966) has assumed that the congruences are twist-free, that is, hypersurface orthogonal. Such a restriction is not necessary and will now be relaxed. In fact it has been proved (Griffiths, 1975b) that if the two congruences can be affinely parametrized simultaneously then the twist is necessarily nonzero. Now consider equation (3.1) in which ρ is complex and σ and ϵ zero:

$$D\rho = \rho^2 + \Phi_0\bar{\Phi}_0 \tag{5.2}$$

If the term $\Phi_0\bar{\Phi}_0$ is taken as constant, then the solutions of this equation will have the phase portrait given in Figure 3. If the magnitude of one wave is approximately constant, then Figure 3 describes the relation between the contraction and the twist of the rays of the opposing wave. The phase portrait contains two critical points. These correspond to solutions with zero contrac-

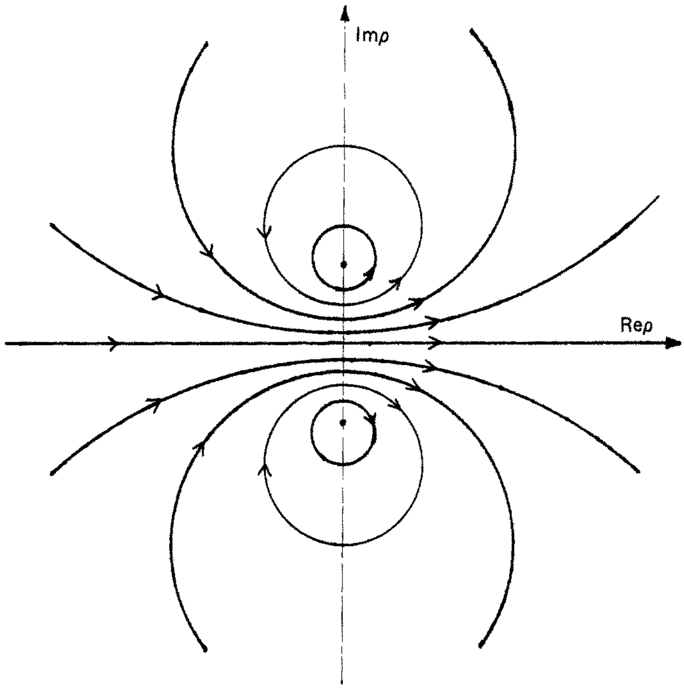

Figure 3—Phase portrait for solutions of equation (5.2) with $\Phi_0\bar{\Phi}_0$ a constant. This describes the relation between the contraction and the twist of the congruences in the interaction region.

tion and constant nonzero twist. An exact solution corresponding to these points has been given previously (Griffiths, 1976). It describes two electromagnetic waves of constant magnitude passing through each other, the twist of one wave being proportional to the energy density of the other. The equivalent solution for twisting noncontracting gravitational waves has also been given (Griffiths, 1975a). It can be seen from the phase portrait that these solutions appear to be stable with respect to perturbations in the twist and contraction, and therefore are of particular physical interest.

The other solution to which particular interest is attached is that with zero twist. This solution in the phase portrait indicates that the congruence will start with infinite expansion (a big bang), which then decreases to zero, and then continues with ever increasing contraction to a final singularity. This describes a case of continuous focusing. An exact solution corresponding to this case will be given in the following section of this paper. Obviously all such solutions are unstable with respect to small perturbations in the twist. Any perturbation would cause an increasing twist that would halt the contraction of the waves and cause them again to expand.

6. *New Solution for the Twist-Free Case*

In this section the twist-free case, when ρ is real, is considered. The interaction region is one of continuous focusing. If the waves are initially expanding as from some source, then the expansion is slowed down, stopped, and if the interaction continues, the rays are then subject to an ever increasing contraction. A general solution describing this situation would be extremely difficult to obtain. However, there is one comparatively simple case for which a solution has been obtained. This is now given.

One particular case of continuously focusing waves is that in which the two waves follow the same paths, but in opposite directions. This situation may be thought of as follows: Consider two sources of electromagnetic radiation a fixed distance apart. The radiation from each source causes the radiation from the other to focus. If the sources are a certain critical distance apart, it is possible that rays emitted from one source actually focus at the other. The solution given below corresponds to such a situation. Obviously this is a very restricted solution, and is only possible for a certain polarization of the two waves. Nevertheless it is an instructive example of the differences between the general relativistic and classical field theories.

Since the two waves follow the same paths in opposite directions, the expansion of the rays from one source at any point equals the contraction of the rays from the other. Similarly the shear and rotation of one null congruence equal minus those of the other. It is therefore possible to scale the tetrad so that

$$\mu = \rho, \qquad \lambda = \bar{\sigma}, \qquad \pi = \bar{\tau}$$

In the following solution the shear of the congruences is necessarily nonzero, and therefore the waves do not satisfy the criterion for independence discussed above. It has already been assumed that the rays are tangent to twist-free null geodesics so that

$$\rho = \bar{\rho}, \qquad \kappa = 0, \qquad \nu = 0, \qquad \Phi_1 = 0$$

The magnitudes of the two waves are now the same at any point and it is possible to assume that

$$\Phi_0 = \bar{\Phi}_2$$

This is also an assumption about the polarization of the two waves, and it immediately implies that

$$\tau = \bar{\alpha} + \beta$$

It is now possible to use the remaining tetrad freedom to put

$$2\alpha = \bar{\tau}, \qquad \epsilon = \bar{\epsilon}, \qquad \gamma = \bar{\gamma}$$

Then with the further assumptions that $\epsilon = \gamma$ and that $\tau \neq 0$ the field equations reduce to

$$D\rho = 2\rho^2 + \tau\bar{\tau} - 2\rho\epsilon \qquad D\rho + \Delta\rho = 0, \qquad \delta\rho = 0 \tag{6.1}$$

$$D\sigma = 2\rho\sigma + \tau^2 - 2\sigma\epsilon + \Phi_0{}^2 \qquad D\sigma + \Delta\sigma = 0, \qquad \delta\sigma = 0 = \bar{\delta}\sigma \tag{6.2}$$

$$D\tau = 3\rho\tau + \sigma\bar{\tau} \qquad D\tau + \Delta\tau = 0, \qquad \delta\tau = 0 = \bar{\delta}\tau \tag{6.3}$$

$$2D\epsilon = \rho^2 - \sigma\bar{\sigma} + 3\tau\bar{\tau} - 4\epsilon^2 \qquad D\epsilon + \Delta\epsilon = 0, \qquad \delta\epsilon = 0 \tag{6.4}$$

$$D\Phi_0 = (\rho - 2\epsilon)\Phi_0 - \sigma\bar{\Phi}_0 \qquad D\Phi_0 + \Delta\Phi_0 = 0, \qquad \delta\Phi_0 = 0 = \bar{\delta}\Phi_0 \tag{6.5}$$

where

$$\Psi_0 = \bar{\Psi}_4 = \tau^2 - 4\sigma\epsilon + \Phi_0{}^2$$

$$\Psi_1 = \bar{\Psi}_3 = \rho\tau - \sigma\bar{\tau}$$

$$\Psi_2 = \bar{\Psi}_2 = \rho^2 - \sigma\bar{\sigma}$$

and the condition

$$\Phi_0\bar{\Phi}_0 = \rho^2 - \sigma\bar{\sigma} + \tau\bar{\tau} - 4\rho\epsilon$$

must be satisfied. The Bianchi identities have already been satisfied.

Now since $\kappa = 0$ and $\rho = \bar{\rho}$, l_μ and n_μ can be written as scalar multiples of gradients. Put

$$l_\mu = B^{-1}u_{,\mu}, \qquad n_\mu = A^{-1}v_{,\mu}$$

and label coordinates $u = x^0, v = x^1, x = x^2, y = x^3$. Then the complete tetrad may be given by

$$l^\mu = A\delta_1{}^\mu + X^2\delta_2{}^\mu + X^3\delta_3{}^\mu$$

$$n^\mu = B\delta_0{}^\mu + Y^2\delta_2{}^\mu + Y^3\delta_3{}^\mu$$

$$m^\mu = \xi^2\delta_2{}^\mu + \xi^3\delta_3{}^\mu$$

It is in fact possible here to put $B = A$, $\xi^2 = 1$, and $\xi^3 = i$. Then putting $s = v - u$, it can be seen that ρ, σ, τ, ϵ, and Φ_0 are all functions of s only. The metric equations now require that

$$A = A(s)$$

$$X^2 = P^2(s) - (\rho + \sigma)x$$

$$X^3 = P^3(s) - (\rho - \sigma)y$$

$$Y^2 = R^2(s) + (\rho + \sigma)x$$

$$Y^3 = R^3(s) + (\rho - \sigma)y$$

where

$$DA = -2\epsilon A \tag{6.6}$$

$$D(P^2 + R^2) = -(\rho + \sigma + 2\epsilon)(P^2 + R^2) + 2(\tau + \bar{\tau}) \tag{6.7}$$

$$D(P^3 + R^3) = -(\rho + \sigma + 2\epsilon)(P^3 + R^3) - 2i(\tau - \bar{\tau}) \tag{6.8}$$

The metric can now be obtained from

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu$$

where

$$l_\mu = A^{-1}\delta_\mu{}^0, \qquad n_\mu = A^{-1}\delta_\mu{}^1$$

$$m_\mu = \frac{Y^2 + iY^3}{2A}\delta_\mu{}^0 + \frac{X^2 + iX^3}{2A}\delta_\mu{}^1 - \frac{1}{2}\delta_\mu{}^2 - \frac{i}{2}\delta_\mu{}^3$$

It now remains to solve the set of ordinary differential equations (6.1)–(6.8) where $D = A(s)\,d/ds$. Although no analytic expression has been obtained, the approximate numerical integration of the equations is straightforward. Taking the origin $s = 0$ at the midpoint between the two sources, the initial conditions may be taken as

$$\rho(0) = 0, \qquad \sigma(0) = 0, \qquad |\tau(0)| = |\Phi_0(0)|, \qquad \epsilon(0) = 0$$
$$A(0) = 1, \qquad P^2(0) + R^2(0) = 0, \qquad P^3(0) + R^3(0) = 0$$

and units may be chosen such that $|\Phi_0(0)| = 1$. It is now a simple matter to integrate the equations for increasing values of s. The sources of the radiation will be situated at points $s = \pm s_1$ which are points where ρ becomes infinite and A zero. In the particular case when σ, τ, and Φ_0 are all real and positive, it should be noticed that $2\epsilon = \rho + \sigma$ so that

$$A(s) = 1 - \int_0^s (\rho + \sigma)\,ds$$

and therefore

$$\int_0^{s_1} (\rho + \sigma)\,ds = 1$$

Numerical integration gives s_1 to be approximately equal to 0.56. Thus in a general system of units

$$s_1 = 0.56(8\pi G)^{-1/2}\,c^2[\Phi_0(0)]^{-1}$$

The above is a very interesting solution of Einstein's equations. No such solution is possible in classical Maxwell theory. It is, however, extremely unlikely that it describes any real macroscopic situation. But it has been pointed out to me by Madore that it may be possible that this solution could be used to construct a geon by identifying the two sources. This has not yet been considered.

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